

A TWO DIMENSIONAL DEPTH-INTEGRATED
FINITE ELEMENT COASTAL CIRCULATION MODEL

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ABSTRACT

A vertically-integrated shallow-water model has been developed to address coastal circulation problems. It uses a triangular grid in conjunction with the finite element method. The model accounts for tidal changes, wind drag, lateral in/out flow and other factors. Partial results of a sample computer run for Apra Harbor, Guam, are given. These results have not yet been verified with field data.

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1. INTRODUCTION

Modification of coastal waters associated with construction and operation of facilities can significantly alter the natural physical, chemical and biological processes. Tropical coastal waters, where extensive biological activities take place in the characteristic shallow reef regions, are particularly susceptible to impact by coastal modifications. As the result of accelerated development efforts, even the most remote tropical islands are experiencing the effect of man-made change. The Water and Energy Research Institute of the University of Guam is often challenged by the task of evaluating the consequences of such activities. As an initial step toward gaining insight into the complexities of the problem, methods of assessing the impact on circulation patterns are needed. Analyses of water mass movements and estimates of the fate of dissolved and suspended matters are the fundamental requirements for such studies. They are also essential elements for the sound management of coastal resources.

There are two approaches for studying coastal processes: physical modeling and numerical modeling. Physical models are very useful when properly verified but they are very costly and time-consuming for a small research institution. With the increased accessibility and improved economy of high-speed digital computers, the numerical modeling approach has become a practical tool for simulating coastal processes.

Ideally, three-dimensional modeling is most desirable but uncertainties in boundary conditions and in eddy viscosities prohibit

its application to large-scale problems. Alternatively, the vertically-integrated shallow water formulation is a practical approach for tropical coastal problems provided that its limitations are strictly observed. It is an attempt to simplify a very complex problem by eliminating dependence on the vertical coordinate. The term "shallow water" is used to denote a water mass with little change of the variables over the depth. There are a number of reports describing the shallow water numerical models for circulation and dispersion in well-mixed estuaries and coastal waters. The finite difference models by Leendertse (1967) and Abbot et al. (1973) are particularly well known. These models, however, are proprietary and not available to the public. The finite element model of Wang and Connor (1975) is also well known. However, the model has a reputation of being inefficient and difficult for implementation.

Consequently, the Water and Energy Research Institute at the University of Guam has undertaken an effort to develop a general shallow water numerical model suitable for tropical coastal waters. Three major attributes of the model have been emphasized during the course of development: complete generality for a wide range of applications, computational efficiency, and relative ease of application. Two of the major causes of water mass movement, the astronomical tide and lateral in/out flow, are accounted for by prescribing the boundary conditions. The effect of wind drag is also included for model generality. Lateral variations of density can be included, but their distribution must be specified. An explicit treatment of lateral density variation would require simultaneous solution for the conservation of heat energy and salt balance along

with the momentum and continuity equations, which adds excessive complexities to the problem. The finite element method has been chosen as the numerical scheme for its very flexible grid discretization.

This report is a summary of the development effort of such a numerical model for shallow tropical coastal waters. This study is intended to be only the initial stage of the rather ambitious goal of developing a general tropical coastal circulation and water quality simulation technique to be utilized by engineers as well as by marine biologists. Therefore, the content of this report should be viewed as a chapter in the overall pursuit.

2. FORMULATION

Starting with the fundamentals of mass conservation and force equilibrium, a mathematical formulation for unsteady vertically integrated flow is presented here. The approach is similar to the work by Wang and Connor (1975), Leendertse (1967) and Hansen (1956), although the choice of variables is different. It is intended to include important steps of the development and to account for assumptions and their basis. The numerical values needed in the constitutive expressions are also presented based on the existing methodologies.

2.1. Three-Dimensional Equations

The formulation of the mass and momentum conservation for three-dimensional continuum flow has long been established for an Eulerian framework [see, for example, Daily and Harleman (1966)]. For a Cartesian coordinate system, the mass conservation is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = s \quad (1)$$

where ρ is the density of fluid; t is time; u , v , w are the velocity components in x , y , z directions; and s is the internal source or sink. Equation (1) simply states that the local rate of change in mass within a control volume is the difference between efflux and influx out of and into the control surface.

The force equilibrium for the control volume is given for the horizontal x and y directions as:

$$\begin{aligned} & \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} - \rho f v \\ & = - \frac{\partial p}{\partial x} + \frac{\partial(\tau_{xx})}{\partial x} + \frac{\partial(\tau_{yx})}{\partial y} + \frac{\partial(\tau_{zx})}{\partial z} + \rho m_x \end{aligned} \quad (2)$$

$$\begin{aligned} & \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} + \rho f u \\ & = - \frac{\partial p}{\partial y} + \frac{\partial(\tau_{xy})}{\partial x} + \frac{\partial(\tau_{yy})}{\partial y} + \frac{\partial(\tau_{zy})}{\partial z} + \rho m_y \end{aligned} \quad (3)$$

where the z-axis is directed vertically upwards; τ_{ij} are the normal and shear stresses due to molecular viscosity and turbulent momentum transfer; f is the Coriolis parameter ($f = 2 \Omega \sin \phi$, where Ω is the phase velocity of the earth's rotation and ϕ is the latitude of the location for the northern hemisphere); p is the pressure; and m_x and m_y are the internal sources of momentum. The stresses τ_{ij} , $i, j = 1, 2, 3$, are defined as usual, the first index denoting the normal direction of the face on which the stress acts and the second, the direction of the stress. The subscript i, j, k is the standard tensor index, the 1, 2, 3 directions being equivalent to x, y, z . In the horizontal momentum equations, the vertical component of the Coriolis force is dropped out by an order of magnitude comparison.

The vertical force equilibrium is given by

$$\begin{aligned} & \frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} + 2\rho\omega_x v - 2\rho\omega_y u \\ & = - \frac{\partial p}{\partial z} - \rho g + \frac{\partial(\tau_{xz})}{\partial x} + \frac{\partial(\tau_{yz})}{\partial y} + \frac{\partial(\tau_{zz})}{\partial z} \end{aligned} \quad (4)$$

where ω_x and ω_y are the x and y components of the earth's rotation. An order of magnitude comparison applied to Equation (4) leaves only the gravity and pressure terms as significant, yielding the hydrostatic condition:

$$\frac{\partial P}{\partial z} = -\rho g \quad (5)$$

At the free surface, the kinematic condition applies, i.e.,

$$\left. \frac{d(\eta-z)}{dt} \right|_{z=\eta} = \left[\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} - w \right]_{z=\eta} = 0 \quad (6)$$

in which the effects of rainfall and evaporation are ignored.

As shown in Figure 1, $\eta = \eta(x, y, t)$ is the free surface displacement. At the bottom, where $z = -h(x, y)$, the impermeable condition is

$$\left. \frac{d(z+h)}{dt} \right|_{z=-h} = \left[u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + w \right]_{z=-h} = 0 \quad (7)$$

where $h = h(x, y)$ is the mean water depth as defined in Figure 1.

For vertical lateral boundaries, the continuity of flow is required:

$$u_i n_i \Big|_{\text{int.}}^{\text{ext.}} = 0, \quad i = 1, 2 \quad (8)$$

$$l_{ijk} u_i n_j \left| \begin{array}{l} \text{ext.} \\ \text{int.} \end{array} \right. = 0, \quad i, j, k = 1, 2, 3 \quad (9)$$

where l_{ijk} is the permutation tensor, $l_{123} = l_{312} = l_{231} = 1$, $l_{132} = l_{213} = l_{321} = -1$ and all other elements are zero; n_i is a unit normal vector directed outward from the area of interest; and int., ext. are points just inside and outside of the boundary. Equations (8) and (9) state that normal and tangential velocities just outside and inside the boundary must be equal.

On all the boundaries force equilibrium must be satisfied.

At the free surface,

$$\tau_y^s + p \frac{s \partial \eta}{\partial x} = \left[p \frac{\partial \eta}{\partial x} - \tau_{xx} \frac{\partial \eta}{\partial x} - \tau_{xy} \frac{\partial \eta}{\partial y} - \tau_{zx} \right]_{z=\eta} \quad (10)$$

$$\tau_y^s + p \frac{s \partial \eta}{\partial y} = \left[p \frac{\partial \eta}{\partial y} - \tau_{xy} \frac{\partial \eta}{\partial x} - \tau_{yy} \frac{\partial \eta}{\partial y} - \tau_{zy} \right]_{z=\eta} \quad (11)$$

$$-p^s + \tau_{xx} \frac{s \partial \eta}{\partial x} + \tau_{yy} \frac{s \partial \eta}{\partial y} = \left[-p - \tau_{xz} \frac{\partial \eta}{\partial x} - \tau_{yz} \frac{\partial \eta}{\partial y} + \tau_{zz} \right]_{z=\eta} \quad (12)$$

and at the bottom [$z = -h(x, y)$],

$$\tau_x^b - p \frac{b \partial h}{\partial x} = \left[- (p - \tau_{xx}) \frac{\partial h}{\partial x} + \tau_{yx} \frac{\partial h}{\partial y} + \tau_{xz} \right]_{z=-h} \quad (13)$$

$$\tau_y^b - p \frac{b \partial h}{\partial y} = \left[\tau_{xy} \frac{\partial h}{\partial x} - (p - \tau_{yy}) \frac{\partial h}{\partial y} + \tau_{zy} \right]_{z=-h} \quad (14)$$

$$-p^b - \tau_x^s \frac{\partial h}{\partial x} - \tau_y^s \frac{\partial h}{\partial y} = \left[-p + \tau_{xz} \frac{\partial h}{\partial x} + \tau_{yz} \frac{\partial h}{\partial y} + \tau_{zz} \right]_{z=-h} \quad (15)$$

In Equations (10) through (15), superscripts s and b denote surface and bottom respectively.

On lateral boundaries, the stresses must be continuous, i.e.,

$$[\text{normal stress}]_{\text{int.}}^{\text{ext.}} = 0 \quad (16)$$

$$[\text{tangential stress}]_{\text{int.}}^{\text{ext.}} = 0 \quad (17)$$

Finally, the initial flow field must be known:

$$u_i \Big|_{t=0} = u_{0i}, \quad i = 1, 2, 3 \quad (18)$$

For a complex arbitrary geometry, the three-dimensional problem as formulated is very difficult to solve. Numerical solutions not only require excessive computer resources but are obstructed by the difficulties in incorporating the proper boundary conditions. In shallow coastal areas where properties of the water column are nearly homogeneous through the depth, the problem can be significantly simplified by removing the explicit dependence on the vertical coordinate. In the following subsection, this process is formally treated in detail.

2.2 Vertical Integration

In shallow coastal waters, the dependence of flow on depth is not very profound and vertically integrated equations may adequately describe the current field. The vertically integrated approach deals with the mass transport through any cross section. However, it yields no information on the vertical velocity profile. As a first step, the water density is assumed constant in the vertical direction, i.e. $\rho = \rho(x, y, t)$. The shallowness of water also implies that the vertical components of velocity and acceleration are relatively small.

The governing equations are integrated over the entire depth to eliminate the dependence on the z-direction. The mass conservation equation [Equation (1)] is integrated as follows:

$$\int_{-h}^{\eta} \frac{\partial \rho}{\partial t} dz + \int_{-h}^{\eta} \frac{\partial(\rho u)}{\partial x} dz + \int_{-h}^{\eta} \frac{\partial(\rho v)}{\partial y} dz + \int_{-h}^{\eta} \frac{\partial(\rho w)}{\partial z} dz = \int_{-h}^{\eta} s dz \quad (19)$$

Applying the Leibnitz's rule, it follows that

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{-h}^{\eta} \rho dz - \rho \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^{\eta} \rho u dz - \rho u \Big|_{\eta} \frac{\partial \eta}{\partial x} + \rho u \Big|_{-h} \frac{\partial(-h)}{\partial x} \\ & + \frac{\partial}{\partial y} \int_{-h}^{\eta} \rho v dz - \rho v \Big|_{\eta} \frac{\partial \eta}{\partial y} + \rho v \Big|_{-h} \frac{\partial(-h)}{\partial y} + \rho w \Big|_{\eta} - \rho w \Big|_{-h} = \rho Q_s \quad (20) \end{aligned}$$

Finally, applying the kinematic conditions on the bottom and the surface yields

$$\frac{\partial(\rho H)}{\partial t} + \frac{\partial(\rho Q_x)}{\partial x} + \frac{\partial(\rho Q_y)}{\partial y} = \rho Q_s \quad (21)$$

where $H = \int_{-h}^n dz = h + \eta$ (22)

$$Q_x = \int_{-h}^n u dz \quad (23)$$

$$Q_y = \int_{-h}^n v dz \quad (24)$$

$$Q_s = \text{internal source/sink flux} \quad (25)$$

The internal source/sink, Q_s , represents the net rate of volume added per unit area and is included here for generality (i.e., inclusion of precipitation, power plant intake/discharge etc.).

The integration of the x- momentum equation [Equation (2)] is analogously performed to yield

$$\begin{aligned} & \frac{\partial(\rho Q_x)}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^n (\rho u^2) dz + \frac{\partial}{\partial y} \int_{-h}^n (\rho uv) dz - \rho f Q_y + \frac{\partial}{\partial x} \int_{-h}^n p dz \\ & - \frac{\partial}{\partial x} \int_{-h}^n \tau_{xx} dz - \frac{\partial}{\partial y} \int_{-h}^n \tau_{yx} dz - \tau_x^s + \tau_x^b - \rho \int_{-h}^n m_x dz \\ & - p^s \frac{\partial H}{\partial x} - p^b \frac{\partial h}{\partial x} = 0 \end{aligned} \quad (26)$$

Invoking the Boussinesque approximation [see, for example, Phillips (1966)] but retaining the real density for the pressure term, Equation (26) is simplified to

$$\begin{aligned} \frac{\partial Q_x}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^n u^2 dz + \frac{\partial}{\partial y} \int_{-h}^n uv dz - fQ_y + \frac{\partial}{\partial x} (F_p - F_{xx}) \\ - \frac{\partial}{\partial y} F_{yx} + \frac{\tau_x^s - \tau_x^b}{\rho_0} - M_x - \frac{p^s}{\rho_0} \frac{\partial H}{\partial x} - g\eta \frac{\partial h}{\partial x} - \frac{\Delta\rho}{\rho_0} gH \frac{\partial h}{\partial x} = 0 \end{aligned} \quad (27)$$

in which

$$F_p = \rho_0 g h \eta + \frac{1}{2} \rho_0 g \eta^2 + \frac{1}{2} \Delta\rho g H^2 + p^s H \quad (28)$$

$$F_{xx} = \frac{1}{\rho_0} \int_{-h}^n \tau_{xx} dz \quad (29)$$

$$F_{yx} = \frac{1}{\rho_0} \int_{-h}^n \tau_{yx} dz \quad (30)$$

$$M_x = \int_{-h}^n m_x dz \quad (31)$$

with

$$\rho = \rho_0 + \Delta\rho ;$$

$$\rho_0 = \text{mean density} ;$$

and

$$\Delta\rho = \text{local deviation (instantaneous)}$$

Similarly, the y-momentum equation becomes,

$$\begin{aligned} \frac{\partial Q_y}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^n uv dz + \frac{\partial}{\partial y} \int_{-h}^n v^2 dz + fQ_x + \frac{\partial}{\partial y} (F_p - F_{yy}) \\ - \frac{\partial}{\partial x} F_{xy} + \frac{\tau_y^s - \tau_y^b}{\rho_0} - M_y - \frac{p^s}{\rho_0} \frac{\partial H}{\partial y} - g\eta \frac{\partial h}{\partial y} - \frac{\Delta\rho}{\rho_0} gH \frac{\partial h}{\partial y} = 0 \end{aligned} \quad (32)$$

where

$$F_{xy} = \frac{1}{\rho_0} \int_{-h}^n \tau_{yx} dz \quad (33)$$

$$F_{yy} = \frac{1}{\rho_0} \int_{-h}^n \tau_{yy} dz \quad (34)$$

$$M_y = \int_{-h}^n m_y dz \quad (35)$$

2.3 Boundary Conditions

Definition of correct boundary conditions is the most critical part of the formulation process. Insufficient boundary information usually results in non-unique solutions while too much may lead to no solutions at all. Although there is no mathematical proof of existence and uniqueness of solutions to Navier-Stokes equations, the present problem is simple enough to deduce necessary boundary conditions by engineering intuition.

By analogy to the Newton's second law governing the motion of a single particle, it is seen that the generalized requirement to the present problem is to specify an initial condition and the force or the discharge on the physical boundaries.

The initial condition is expressed as

$$\left. \begin{aligned} U(x, y) &= U_0(x, y) \\ V(x, y) &= V_0(x, y) \\ H(x, y) &= H_0(x, y) \end{aligned} \right\} \text{at } t = 0 \quad (36)$$

On the physical boundaries two alternatives are possible as mentioned earlier. On boundaries where discharge is specified the

normal and tangential components must be prescribed, i.e.,

$$Q_n = \alpha_{nx}Q_x + \alpha_{ny}Q_y : \text{prescribed} \quad (37)$$

$$Q_s = \alpha_{ny}Q_x - \alpha_{nx}Q_y : \text{prescribed} \quad (38)$$

where α is the direction cosine, α_{nx} being $\cos(n, x)$ and $\alpha_{ny} = \cos(n, y)$.

On the boundaries where the external force is specified, the normal and tangential specific forces (force per unit width per unit density) are:

$$F_{nn} = -F_p + \alpha_{nx}^2 F_{xx} + \alpha_{ny}^2 F_{yy} + 2\alpha_{nx}\alpha_{ny}F_{xy} : \text{prescribed} \quad (39)$$

$$F_{ns} = (\alpha_{nx}^2 - \alpha_{ny}^2)F_{xy} + \alpha_{nx}\alpha_{ny}(F_{yy} - F_{xx}) : \text{prescribed} \quad (40)$$

The free surface elevation can be found by the continuity equation [Equation (21)].

2.4 Internal Stresses

In order to close the formulation, the internal stress terms τ_{xx} , τ_{yy} , and $\tau_{xy} = \tau_{yx}$ must be expressed as functions of the integrated flow variables. As with the common practice for the closure of turbulent flow problems [see, for example, Schlichting (1968)], the stresses are represented by means of the eddy viscosity concept. Although this approach is based on a somewhat imprecise foundation, it does provide a convenient expression which simulate the real effects of internal friction and transfer of momentum.

Following the concept of the Reynolds stress for three-dimensional turbulent flow [see, for example, Daily and Harleman (1966)], the ensemble averaging of an instantaneous velocity product is introduced. Assuming the flow field is essentially random, the instantaneous local velocity components U, V may be expressed as

$$U = u + u' = (\bar{u} + \bar{u}') + (u^\circ + u'^\circ) \quad (41)$$

$$V = v + v' = (\bar{v} + \bar{v}') + (v^\circ + v'^\circ) \quad (42)$$

where u, v are ensemble averages; u' and v' are fluctuation components whose ensemble averages are zero by definition; \bar{u} and \bar{v} are vertically averaged values of u, v ; \bar{u}' and \bar{v}' are vertical average values of u', v' ; u° and v° are vertical deviations of u, v from \bar{u}, \bar{v} ; and u'° and v'° are vertical deviations of u', v' from \bar{u}', \bar{v}' . The product $U \cdot V$ is, therefore, written as

$$\begin{aligned} U \cdot V = & \bar{u}\bar{v} + \bar{u}\bar{v}' + \bar{u}v^\circ + \bar{u}v'^\circ + \bar{u}'\bar{v} \\ & + \bar{u}'\bar{v}' + \bar{u}'v^\circ + \bar{u}'v'^\circ + u^\circ\bar{v} + u^\circ\bar{v}' \\ & + u^\circ v^\circ + u^\circ v'^\circ + u'^\circ\bar{v} + u'^\circ\bar{v}' + u'^\circ v'^\circ \end{aligned} \quad (43)$$

Since all terms with just one prime (turbulent fluctuation) are averaged out, the ensemble average of Equation (43) results

$$\begin{aligned} \langle UV \rangle_{\text{ensemble}} = & \bar{u}\bar{v} + \bar{u}v^\circ + \bar{u}'\bar{v}' + \bar{u}'v'^\circ \\ & + u^\circ\bar{v} + u^\circ v^\circ + u'^\circ\bar{v}' + u'^\circ v'^\circ \end{aligned} \quad (44)$$

Further, averaging over the depth yields

$$\int_{-h}^n \langle UV \rangle_{\text{ensemble}} dz = \frac{1}{H} (Q_x Q_y) + \int_{-h}^n \langle u'v' + u'^{\circ}v'^{\circ} \rangle_{\text{ensemble}} dz + \int_{-h}^n u^{\circ}v^{\circ} dz \quad (45)$$

The second term on the right-hand side represents the turbulent Reynolds stress and the other two terms arise from the momentum transfer due to the vertical velocity shear. By analogy to the Prandtl's mixing length theory [see Schlichting (1968) for detail], the following functional relationship can be deduced:

$$F_{x_i x_j} = \int_{-h}^n \left\{ \frac{\tau_{x_i x_j}^v}{\rho_0} - \langle (u'_i u'_j) \rangle_{\text{ensemble}} - u_i^{\circ} u_j^{\circ} \right\} dz = \epsilon_{ij} \left(\frac{\partial Q_j}{\partial x_i} + \frac{\partial Q_i}{\partial x_j} \right) \quad i, j = 1, 2 \quad (46)$$

In the above expression, ϵ_{ij} is a symmetric apparent eddy viscosity coefficient matrix which is a function of flow condition.

For the anisotropic situation, the directional relationship between ϵ_{ij} and the local mean current is

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \quad (47)$$

with the major principal axis of ϵ_{ij} in the direction of the local mean current. The ϵ_{ij} in the global x-y coordinate is expressed by rotation as

$$\epsilon_{ij \text{ global}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (48)$$

where superscript T indicates transpose and θ is the angle from global x-axis to the local mean current direction.

As to what values of ϵ_{ij} should actually be used must unfortunately depend on experience or trial and error since the explicit functional representation is not possible. Values ranging from 10^0 to 10^5 m²/sec are frequently cited for the principal direction of ϵ_{ij} and the ratio of ϵ_1 to ϵ_2 is normally in the range of 10 to 60 [Wada, (1972)]. In numerical experiments conducted during the present development, values of up to 10^4 m²/sec have not affected the results appreciably. In the computational point of view, the inclusion of $F_{x_i x_j}$ is a significant advantage since it introduces a damping effect for short wave ripple noise.

2.5 Bottom Friction

Previous modeling efforts for vertically integrated flow [Leendertse (1967), for example] have shown that the friction due to the bottom shear can be adequately represented by a quadratic expression:

$$\tau_x^b = C_f \rho (Q_x^2 + Q_y^2)^{\frac{1}{2}} \frac{Q_x}{H^2} \quad (49)$$

$$\tau_y^b = C_f \rho (Q_x^2 + Q_y^2)^{\frac{1}{2}} \frac{Q_y}{H^2} \quad (50)$$

where C_f is a non-dimensional friction factor.

The roughness coefficients of Manning or Chezy are frequently and successfully used to relate the value of C_f to bottom roughness although they were originally derived from measurements of steady flow in pipes and channels. For fixed roughness, C_f relates to

Manning's n and Chezy's C as follows:

$$C_f = \frac{n^2}{H^3} g \quad (\text{Manning}) \quad (51)$$

$$C_f = \frac{g}{C^2} \quad (\text{Chezy}) \quad (52)$$

Values of Manning's n are only valid for fully developed turbulent flow which is usually the case in coastal waters. For other flow regimes, the use of other approaches, such as Darcy-Weisbach's f_{DW} , may be necessary.

A point of caution must be given here with respect to the limitation of the vertically integrated approach. Since the bottom shear stress is related only to discharge per unit width regardless of the velocity profile, the bottom friction cannot be represented properly for flows with profound vertical variation such as might occur for a strong wind-driven surface current with return current underneath. This is the most significant limitation of the vertically integrated approach. However, the initial assumption of vertical homogeneity is considered to exclude such situations.

2.6 Wind Stress

The wind stress on the water surface is commonly represented by the following semi-empirical expression:

$$\tau_s = \rho_a C_D U_{10}^2 \quad (53)$$

where ρ_a is the density of air ($\sim 1.2 \text{ kg/m}^3$); U_{10} is the wind speed at 10 m above the surface; and C_D is the wind drag coefficient.

The evaluation of C_D for water surface is not a simple matter because of the surface deformation and wave formation. Experimental values cited in the literature [Denman and Miyake (1973); Heaps

(1969); Whitaker, et al. (1973); Wu (1969); and others] scatter widely from site to site and for different wind conditions. Among numerous proposed empirical expressions, Wu's relationships based on extensive ocean data seem to give the best fit. Wu's curve fitting is divided into three flow regimes as

$$C_D = \begin{cases} 1.25/U_{10}^{1/2} \cdot 10^{-3} & \text{for } U_{10} \leq 1\text{m/sec} \\ 0.5 \cdot U_{10}^{1/2} \cdot 10^{-3} & \text{for } 1 < U_{10} < 15\text{m/sec} \\ 2.6 \cdot 10^{-3} & \text{for } U_{10} \geq 15\text{m/sec} \end{cases} \quad (54)$$

Even with the Wu's best fit curve, data scatter with a factor of 2. However, considering the complexities of air-sea interaction, such an error is unavoidable and tolerable.

2.7 Summary of Closed Formulation

For convenience, all the pertinent equations for the vertically integrated formulation are summarized below in a closed form.

Conservation of mass:

$$\frac{\partial H}{\partial t} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = Q_I \quad (55)$$

x-momentum:

$$\begin{aligned} & \frac{\partial Q_x}{\partial t} + \frac{\partial (\bar{U}Q_x)}{\partial x} + \frac{\partial (\bar{U}Q_y)}{\partial y} - fQ_y + \frac{\partial (F_p - F_{xx})}{\partial x} \\ & - \frac{\partial F_{yx}}{\partial y} + \frac{1}{\rho_0} (\tau_x^s - \tau_x^b) - M_x - \frac{1}{\rho_0} (p^s \frac{\partial H}{\partial x} + \Delta \rho g H \frac{\partial h}{\partial x}) \\ & - g h \frac{\partial h}{\partial x} = 0 \end{aligned} \quad (56)$$

y-momentum:

$$\begin{aligned} & \frac{\partial Q_y}{\partial t} + \frac{\partial(\bar{v}Q_x)}{\partial x} + \frac{\partial(\bar{v}Q_y)}{\partial y} + fQ_x + \frac{\partial(F_p - F_{yy})}{\partial y} \\ & - \frac{\partial F_{xy}}{\partial x} + \frac{1}{\rho_0} (\tau_y^s - \tau_y^b) - M_y - \frac{1}{\rho_0} (p^s \frac{\partial H}{\partial y} + \Delta \rho g H \frac{\partial h}{\partial y}) \\ & - g_n \frac{\partial h}{\partial y} = 0 \end{aligned} \quad (57)$$

with the constitutive relations:

$$F_p = gh_n + \frac{1}{2} g_n^2 + \frac{1}{2} \frac{\Delta \rho}{\rho_0} g H^2 + \frac{p^s}{\rho_0} H \quad (58)$$

$$F_{x_i x_j} = \epsilon_{ij} \left(\frac{\partial Q_j}{\partial x_i} + \frac{\partial Q_i}{\partial x_j} \right), \quad i, j = 1, 2 \quad (59)$$

Initial conditions:

$$\left. \begin{aligned} H(x, y) &= H_0(x, y) \\ Q_x(x, y) &= Q_{x_0}(x, y) \\ Q_y(x, y) &= Q_{y_0}(x, y) \end{aligned} \right\} \text{at } t = 0 \quad (60)$$

Boundary conditions:

$$\left. \begin{aligned} Q_n &= \alpha_{nx} Q_x + \alpha_{ny} Q_y \\ Q_s &= -\alpha_{ny} Q_x + \alpha_{nx} Q_y \end{aligned} \right\} \text{prescribed} \quad (61)$$

$$\text{or } \left. \begin{aligned} F_{nn} &= -F_p + \alpha_{nx}^2 F_{xx} + \alpha_{ny}^2 F_{yy} + 2\alpha_{nx}\alpha_{ny} F_{xy} \\ F_{ns} &= (\alpha_{nx}^2 - \alpha_{ny}^2) F_{xy} + \alpha_{nx}\alpha_{ny} (F_{yy} - F_{xx}) \end{aligned} \right\} \text{prescribed} \quad (62)$$

where $\alpha_{nx} = \cos(n, x)$ and $\alpha_{ny} = \cos(n, y)$

Bottom shear stress:

$$\begin{aligned}\tau_x^b &= C_f \rho (Q_x^2 + Q_y^2)^{\frac{1}{2}} \frac{Q_x}{H^2} \\ \tau_y^b &= C_f \rho (Q_x^2 + Q_y^2)^{\frac{1}{2}} \frac{Q_y}{H^2}\end{aligned}\tag{63}$$

with expression of C_f given by Equation (51) or (52)

Wind stress:

$$\tau^s = \rho_a C_D U_{f_0}^2\tag{64}$$

where C_D is given by Equation (54)

3. METHOD OF NUMERICAL SOLUTION

The problem presented in the preceding chapter is of such complexities that the solution can be obtained only by a numerical approximation technique. The almost identical shallow water problems have been dealt with in the past by many mathematical modelers using the finite difference method with a variety of assumptions and time-integration schemes (Hansen, 1956; Reid and Bodine, 1968; Heaps, 1969; Simon, 1971; Abbot et al., 1973; Leendertse, 1973 and others) and by a few using the finite element method (Gallagher et al., 1973; Taylor and Davis, 1972; Norton et al., 1973; Wang and Connor, 1975). Although many of these previous numerical modeling efforts were quite successful at least for site specific cases, none of them made program listings or user's manuals available to the public for quite understandable reasons.

The finite difference method (FDM) is a pointwise approximation to the governing equations by replacing the continuous derivatives by discrete difference expressions. Its application to the present shallow water problem would be rather straight forward, although extremely tedious, since the numerical behavior of FDM is relatively well understood. In addition, the formulation of FDM does not require advanced mathematical knowledge except simple differential calculus and can be easily understood by intuitive physical analogy. However, when irregular complex geometry or unusual specification of boundary conditions is encountered, the FDM technique exhibits its inherent inflexibility. The finite element method (FEM), on the other hand gives a piecewise approximation to the governing equations and a solution domain can be analytically modeled or

approximated by replacing it with an assemblage of discrete elements. Since these elements can be put together in a variety of ways, they can be used to represent exceedingly complex shapes. Furthermore, FEM is inherently advantageous in the generalized treatment of various boundary conditions because of its integral formulation. Although FEM requires somewhat more complex mathematical treatment than FDM, the possibility of using arbitrary grid schematizations and the superior handling of boundaries make this technique particularly attractive for the present purpose of developing a general shallow water circulation model.

The rigorous mathematical foundation, in particular the integral formulation, on which the finite element method is based, is best left to the standard textbooks (Zienkewicz and Cheung, 1967; Zienkewicz, 1971; Connor, 1973; Huebner, 1975), and therefore, only the practical computational aspects are presented in this chapter.

3.1. Integral Form of Governing Equations

Following the standard procedure (see, for example, Connor, 1973), the equation of continuity [Equation 55] is transformed into an integral form as

$$\int_{\Omega} \left[\frac{\partial \tilde{H}}{\partial t} + \frac{\partial \tilde{Q}_x}{\partial x} + \frac{\partial \tilde{Q}_y}{\partial y} - Q_i \right] \delta H \, dA = 0 \quad (65)$$

where Ω is the interior of the domain of interest; \tilde{H} , \tilde{Q}_x and \tilde{Q}_y are approximate functions (trial functions) of H , Q_x and Q_y ; δH is an assumed function (test function).

The momentum equations [Equations 56 and 57] with the boundary conditions [Equations 61 and 62] are transformed as follows:

$$\begin{aligned}
& \int_{\Omega} \left\{ \left[\frac{\partial \tilde{Q}_x}{\partial t} + \frac{\partial}{\partial x} (\bar{u} \tilde{Q}_x) + \frac{\partial}{\partial y} (\bar{v} \tilde{Q}_y) - f \tilde{Q}_y + \frac{1}{\rho_0} (\tau_x^s - \tau_x^b) - \bar{M}_x \right. \right. \\
& \quad \left. \left. - \frac{1}{\rho_0} (p^s \frac{\partial \tilde{H}}{\partial x} + \Delta \rho g \tilde{H} \frac{\partial h}{\partial x}) - g \eta \frac{\partial h}{\partial x} \right] \delta Q_x - [(F_p - F_{xx}) \frac{\partial}{\partial x} (\delta Q_x) \right. \\
& \quad \left. - F_{yx} \frac{\partial}{\partial y} (\delta Q_x)] \right\} dA - \int_{\Gamma_Q} [\alpha_{nx} (F_{xx} - F_p) + \alpha_{ny} F_{yx}] \delta Q_x d\ell \\
& \quad + \int_{\Gamma_F} \alpha_{nx} F_p^* \delta Q_x d\ell + \int_{\Gamma_Q} [\alpha_{nx} Q_n - \alpha_{ny} Q_s - Q_x^*] \delta Q_x d\ell = 0 \quad (66)
\end{aligned}$$

$$\begin{aligned}
& \int_{\Omega} \left\{ \left[\frac{\partial \tilde{Q}_y}{\partial t} + \frac{\partial}{\partial x} (\bar{v} \tilde{Q}_x) + \frac{\partial}{\partial y} (\bar{v} \tilde{Q}_y) + f \tilde{Q}_x + \frac{1}{\rho_0} (\tau_y^s - \tau_y^b) - \bar{M}_y \right. \right. \\
& \quad \left. \left. - \frac{1}{\rho_0} (p^s \frac{\partial \tilde{H}}{\partial y} + \Delta \rho g \tilde{H} \frac{\partial h}{\partial y}) - g \eta \frac{\partial h}{\partial y} \right] \delta Q_y - [(F_p - F_{yy}) \frac{\partial}{\partial y} (\delta Q_y) \right. \\
& \quad \left. - F_{xy} \frac{\partial}{\partial x} (\delta Q_y)] \right\} dA - \int_{\Gamma_Q} [\alpha_{ny} (F_{yy} - F_p)] \delta Q_y d\ell \\
& \quad + \int_{\Gamma_F} \alpha_{ny} F_p^* \delta Q_y d\ell + \int_{\Gamma_Q} [\alpha_{ny} Q_n + \alpha_{nx} Q_s - Q_y^*] \delta Q_y d\ell = 0 \quad (67)
\end{aligned}$$

where the superscript, *, indicates a prescribed boundary value, and for the line integrals over the prescribed force boundary Γ_F and flow boundary Γ_Q the following expressions are used for the x and y components of force and flow:

$$\alpha_{nx} (F_{xx} - F_p) + \alpha_{ny} F_{yx} = F_{nx} \quad (68)$$

$$\alpha_{nx} F_{xy} + \alpha_{ny} (F_{yy} - F_p) = F_{ny} \quad (69)$$

$$\alpha_{nx} Q_n - \alpha_{ny} Q_s = Q_x \quad (70)$$

$$\alpha_{nx} Q_s + \alpha_{ny} Q_n = Q_y \quad (71)$$

A detailed mathematical review of these integral transformation can be found in Wang and Connor (1975).

3.2 Finite Element Formulation

The integral forms of the governing equations are general and can be applied to any consistent type of element. In two spatial dimensions, the linear triangular element is the simplest and best suited for the present objectives. The standard procedure is to develop the functional relationship for a typical element and then assemble the contribution from each element to obtain a system of equations. Following Huebner (1975), the triangular element is based on a linear expansion for the variables, i. e.,

$$\tilde{H} = [\phi] H^e \quad (72)$$

$$\tilde{Q}_x = [\phi] Q_x^e \quad (73)$$

$$\tilde{Q}_y = [\phi] Q_y^e \quad (74)$$

where the superscript e denotes the nodal values, $[\phi]$ is the shape function defined by

$$[\phi] = [\xi_1 \ \xi_2 \ \xi_3] \quad (75)$$

and ξ_i are the natural coordinates of the element defined by

$$\xi_i = \frac{1}{2A} (c_i + b_i x + a_i y) \quad (76)$$

in which $a_i = x_k - x_j$ (77)

$$b_i = y_j - y_k \quad (78)$$

$$c_i = x_j y_k - x_k y_j \quad (79)$$

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad (80)$$

(i, j, k) designates the element nodes (1, 2, 3) numbered counter-clockwise. Thus, a variable, ψ for an element may be expressed in terms of the shape function, ϕ and nodal values $\{\psi^e\}$ as

$$\psi = [\phi] \cdot \{\psi^e\} \quad (81)$$

where $\{\psi^e\}$ is a column vector of nodal values of ψ for the element.

The finite element analog of the integral equation [Equation (65)] for an element is then,

$$\begin{aligned} & \{\delta H^e\}^T \int_A [\phi]^T \left\{ [\phi] \left\{ \frac{\partial H^e}{\partial t} \right\} + \{Q_x^e\} \frac{\partial}{\partial x} [\phi] + \{Q_y^e\} \frac{\partial}{\partial y} [\phi] \right. \\ & \left. - [\phi] \{Q_I^e\} \right\} dA = 0 \end{aligned} \quad (82)$$

where superscript T indicates transpose.

In order to achieve symmetry in the total coefficient matrix, the continuity equation can be modified by multiplying by gh without losing generality as

$$\int_{\Omega} \left\{ gh \frac{\partial H}{\partial t} + gh \frac{\partial Q_x}{\partial x} + gh \frac{\partial Q_y}{\partial y} - gh Q_I \right\} \delta H dA = 0 \quad (83)$$

which results in for an element

$$\begin{aligned} & \{\delta H^e\}^T \int_A [\phi]^T \left\{ g([\phi]\{h^e\})[\phi] \left\{ \frac{\partial H^e}{\partial t} \right\} + g([\phi]\{h^e\})[\phi] \frac{\partial}{\partial x} [\phi] \{Q_x^e\} \right. \\ & \left. + g([\phi]\{h^e\})[\phi] \frac{\partial}{\partial x} [\phi] \{Q_y^e\} - g([\phi]\{h^e\})[\phi] \{Q_I^e\} \right\} dA = 0 \end{aligned} \quad (84)$$

The integral over the entire domain can be obtained by summing the contributions from each element as

$$\begin{aligned} & \int_{\Omega} \left\{ gh \frac{\partial H}{\partial t} + gh \frac{\partial Q_x}{\partial x} + gh \frac{\partial Q_y}{\partial y} - gh Q_I \right\} \delta H dA = 0 \\ & = \sum_{i=1}^{NE} \{\delta H^{e_i}\}^T \int_{A_i} [\phi]^T \left\{ g([\phi]\{h^e\})[\phi] \left\{ \frac{\partial H^e}{\partial t} \right\} + g([\phi]\{h^e\})[\phi] \frac{\partial}{\partial x} [\phi] \{Q_x^e\} \right. \\ & \left. + g([\phi]\{h^e\})[\phi] \frac{\partial}{\partial x} [\phi] \{Q_y^e\} - g([\phi]\{h^e\})[\phi] \{Q_I^e\} \right\} dA = 0 \end{aligned} \quad (85)$$

where NE is the total number of elements in the domain. In a simplified form Equation (86) is expressed as

$$\sum_{i=1}^{NE} \{\delta H^{ei}\}^T \{ [A_h^i] \left\{ \frac{\partial H^{ei}}{\partial t} \right\} + [B_x^i] \{Q_x^{ei}\} + [B_y^i] \{Q_y^{ei}\} - [A_h^i] \{Q_i^{ei}\} \} dA = 0 \quad (86)$$

where

$$\begin{aligned} [A_h^i] &= A_i g [\phi]^T [\phi] \{h\} [\phi] dA \\ &= \frac{gA_i}{60} \begin{bmatrix} 6h_1 + 2h_2 + 2h_3 & 2h_1 + 2h_2 + h_3 & 2h_1 + h_2 + 2h_3 \\ 2h_1 + 2h_2 + h_3 & 2h_1 + 6h_2 + 2h_3 & h_1 + 2h_2 + 2h_3 \\ 2h_1 + h_2 + h_3 & h_1 + 2h_2 + 2h_3 & 2h_1 + 2h_2 + 6h_3 \end{bmatrix} \quad (87) \end{aligned}$$

$$[A^i] = A_i [\phi]^T [\phi] dA = \frac{A_i}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad (88)$$

$$\begin{aligned} [B_x^i] &= A_i g [\phi]^T ([\phi] \{h\}) \frac{\partial}{\partial x} [\phi] dA \\ &= \frac{g[A_i]}{2A_i} \begin{bmatrix} h_1 a_1 & h_1 a_2 & h_1 a_3 \\ h_2 a_1 & h_2 a_2 & h_2 a_3 \\ h_3 a_1 & h_3 a_2 & h_3 a_3 \end{bmatrix} \quad (89) \end{aligned}$$

$$\begin{aligned} [B_y^i] &= A_i g [\phi]^T ([\phi] \{h\}) \frac{\partial}{\partial y} [\phi] dA \\ &= \frac{g[A_i]}{2A_i} \begin{bmatrix} h_1 b_1 & h_1 b_2 & h_1 b_3 \\ h_2 b_1 & h_2 b_2 & h_2 b_3 \\ h_3 b_1 & h_3 b_2 & h_3 b_3 \end{bmatrix} \quad (90) \end{aligned}$$

By carrying out the summation $\sum_{i=1}^{NE}$, yields the following system of equations

$$\{\delta H\}^T \left\{ [A_h] \left\{ \frac{\partial H}{\partial t} \right\} + [B_x] \{Q_x\} + [B_y] \{Q_y\} - [A_h] \{Q_i\} \right\} = 0 \quad (91)$$

Since $\{\delta H\}$ is an arbitrary function, the terms within the large braces must vanish resulting in simultaneous equations for every node point:

$$[A_h] \left\{ \frac{\partial H}{\partial t} \right\} + [B_x] \{Q_x\} + [B_y] \{Q_y\} - [A_h] \{Q_i\} = 0 \quad (92)$$

The equivalent system equations for the momentum equations [Equations (66) and (67)] can be obtained in an analogous manner resulting in

$$[A] \left\{ \frac{\partial Q_x}{\partial t} \right\} - [B_x]^T \{\eta\} + [E_{xx}] \{Q_x\} + [E_{yx}] \{Q_y\} - f[A] \{Q_y\} - [G_x] = 0 \quad (93)$$

$$[A] \left\{ \frac{\partial Q_y}{\partial t} \right\} - [B_y]^T \{\eta\} + [E_{xy}] \{Q_x\} + [E_{yy}] \{Q_y\} + f[A] \{Q_x\} - [G_y] = 0 \quad (94)$$

where $[A]$ is assembled from Equation (88). The terms in the above equations correspond to the contributions of convective acceleration, linear portion of specific force, eddy viscosities and Coriolis force. The remainder is lumped in $[G_x]$ and $[G_y]$. Since eddy viscosities E_{ij} are element property, $[E_{xx}]$, $[E_{yy}]$, and $[E_{xy}] = [E_{yx}]$ are assembled from

$$[E_{ij}^k] = A_k \left([E_{ij}^k] [\phi, i]^T [\phi, j] + \delta_{i1} \delta_{j1} E_{21}^k [\phi, 2]^T [\phi, 2] + \delta_{i2} \delta_{j2} E_{12}^k [\phi, 1]^T [\phi, 1] \right) dA \quad (95)$$

where δ_{ij} is the kronecker δ .

Defining a new nodal variable by coupling Q_x and Q_y as

$$[Q] = \left[\underbrace{Q_x Q_y}_{\text{node 1}} \quad \underbrace{Q_x Q_y}_{\text{node 2}} \quad \dots \quad \underbrace{Q_x Q_y}_{\text{node MAXNOD}} \right] \quad (96)$$

Equations (92), (93) and (94) can be simplified to

$$[A_1]\left\{\frac{\partial H}{\partial t}\right\} + [B_1]\{Q\} = [G_H] \quad (97)$$

$$[A_2]\left\{\frac{\partial Q}{\partial t}\right\} - [B_1]^T\{n\} + [C]\{Q\} + [D]\{Q\} = [G_Q] \quad (98)$$

where $[G_H] = [A_h]\{Q_I\}$ and $[A_1] = [A_h]$. $[A_2]$, $[B_1]$, $[C]$, $[D]$ and $[G_Q]$ are assembled by merging the element matrices according to Equation (96).

Since $\frac{\partial n}{\partial t} \equiv \frac{\partial H}{\partial t}$, Equation (97) becomes

$$[A_1]\left\{\frac{\partial n}{\partial t}\right\} + [B]\{Q\} = [G_H] \quad (99)$$

The partitioned form of Equations (98) and (99) is then

$$\underbrace{\begin{matrix} NM \\ \left\{ \begin{matrix} [A_1] & [0] \\ [0] & [A_2] \end{matrix} \right\} \end{matrix}}_{NM \quad 2 \times NM} \underbrace{\begin{matrix} \left[\begin{matrix} \frac{\partial n}{\partial t} \\ \frac{\partial Q}{\partial t} \end{matrix} \right] \end{matrix}}_{\left[\begin{matrix} \frac{\partial n}{\partial t} \\ \frac{\partial Q}{\partial t} \end{matrix} \right]} + \underbrace{\begin{matrix} [0] & [B] \\ [B]^T & [C]+[D] \end{matrix} \begin{matrix} \{n\} \\ \{Q\} \end{matrix}}_{\begin{matrix} [0] & [B] \\ [B]^T & [C]+[D] \end{matrix} \begin{matrix} \{n\} \\ \{Q\} \end{matrix}} = \underbrace{\begin{matrix} [G_H] \\ [G_Q] \end{matrix}}_{\begin{matrix} [G_H] \\ [G_Q] \end{matrix}} \quad (100)$$

It is now apparent why the modified form of continuity equation [Equation, (83)] has been chosen rather than Equation (82). Complete symmetry or skew-symmetry is achieved in the total system coefficient matrices. This effective approach for the shallow-water equations was first demonstrated by Wang and Connor (1975).

3.3 Time Integration

Efficient and stable numerical integration is one of the most important aspects of the model development. A typical coastal problem would involve hundreds of node points and time integrations over several tidal cycles. The choice of integration scheme depends on the requirements of stability, accuracy, and computational efficiency both in time and memory size. The fully implicit scheme, although desirable in view of stability characteristics, has been discarded since it requires a prohibitively large memory size. Thus, choice has been limited to the variations of the explicit scheme. For its relative simplicity and acceptable stability characteristics, a self starting split time method has been adopted in this study. The method is analogous to the leap-frog method for the finite difference formulation. Using the same notation for Equation 98 and 99, this integration scheme is expressed by

$$[A_1] (\{n\}^{n+\frac{1}{2}} - \{n\}^{n-\frac{1}{2}}) + \Delta t [B] \{Q\}^n = \Delta t [G_H] \quad (101)$$

$$[A_2] (\{Q\}^{n+1} - \{Q\}^n) - \Delta t [B]^T \{n\}^{n+\frac{1}{2}} = \Delta t [G_Q] - \Delta t ([C] - [D]) \{Q\}^n \quad (102)$$

For given initial conditions, $\{n\}^{n-\frac{1}{2}}$ and $\{Q\}^n$, the solution proceeds first for $\{n\}^{n+\frac{1}{2}}$ and then $\{Q\}^{n+1}$ in a time staggered manner.

Explicit stability criteria for finite element formulations such as the present numerical scheme have not been established. An approximate stability measure based on norms was suggested by Wang and Connor

(1975) for such a scheme. It is an analogy with the Courant-Friedrichs-Lewy condition for the finite difference scheme, i.e.

$$\Delta t < \Delta t_{cr} = \frac{s}{\sqrt{2gH}} \quad (103)$$

where Δs is the characteristic node spacing, H is the water depth, and Δt_{cr} is the critical time step for reference. Our experience with the present model, however, indicates that Δt can exceed Δt_{cr} by up to 50% depending on the particular application.

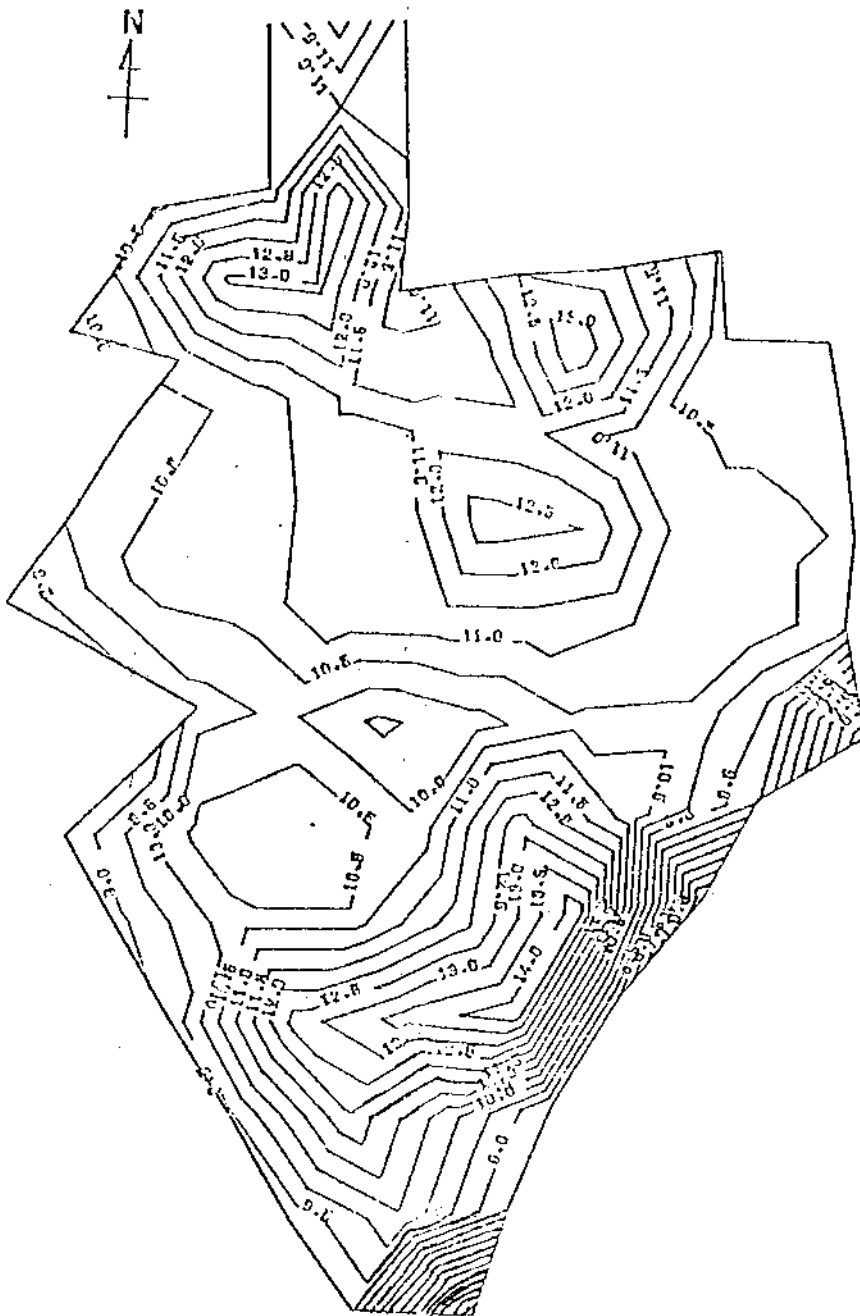


Figure 2. Mean Low Water Depth of Apra Harbor (meters).

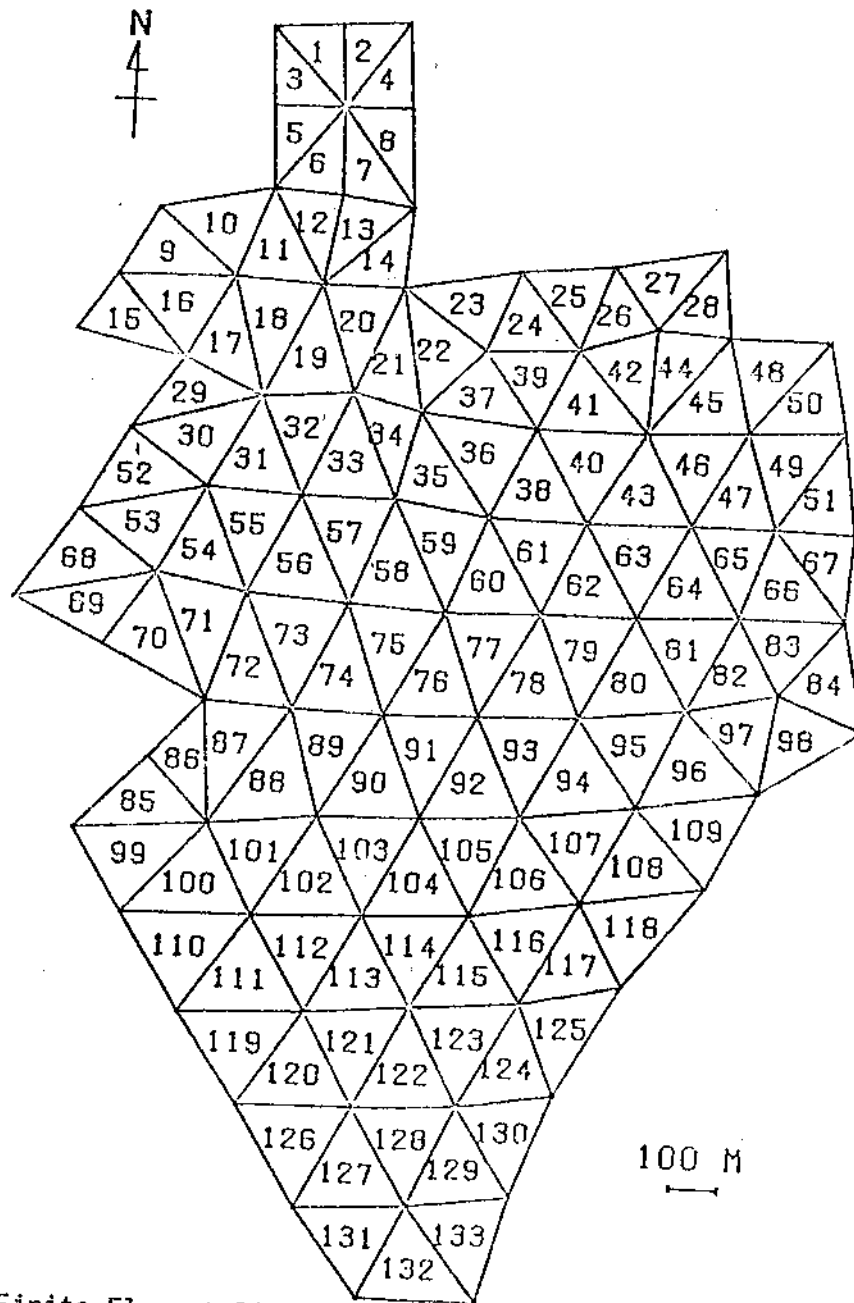


Figure 3. Finite Element Discretization.

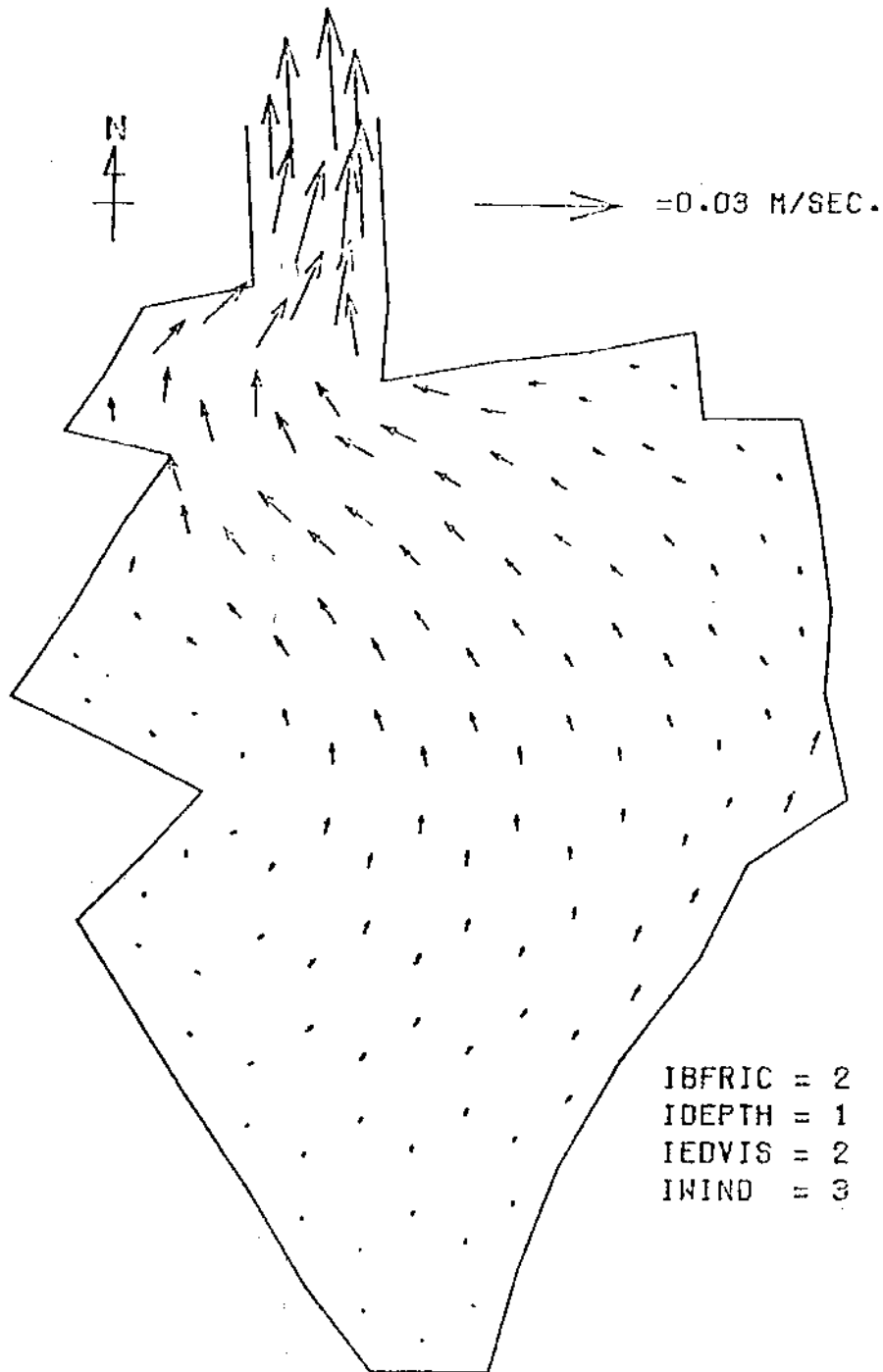


Figure 4. Velocities Averaged Over Depth at t=5,580 sec.

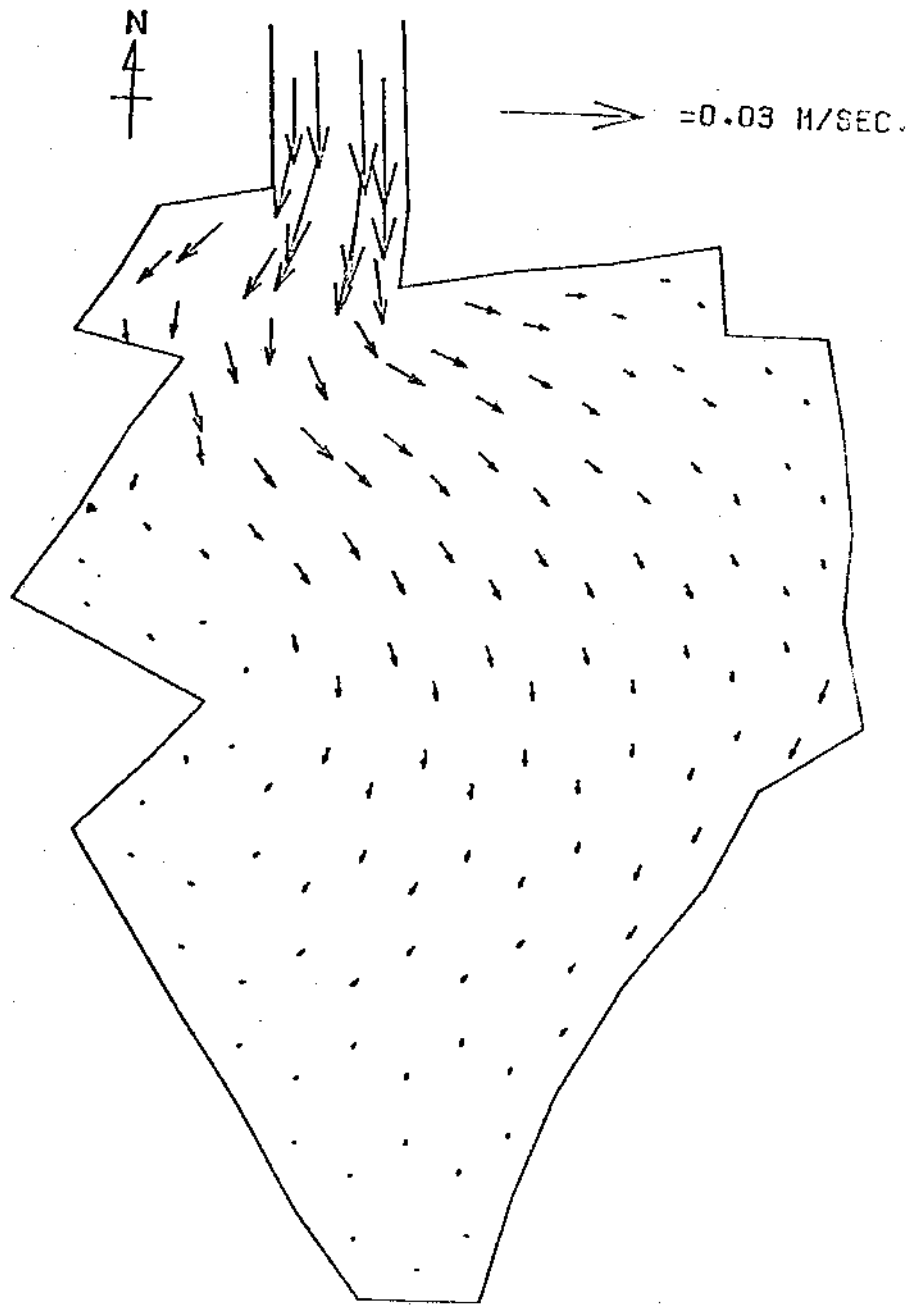


Figure 5. Velocities Averaged Over Depth at $t=27,900 \text{ sec.}$

4. SAMPLE MODEL RUN

Inner Apra Harbor, Guam, was chosen for a sample model run both because of its well-defined physical boundaries and relatively constant depth (Figure 2). The harbor was discretized into 133 elements with 87 modes (Figure 3). The computation was carried out for 1.5 tidal cycles (66,960 seconds) from a low tide with an amplitude of 0.3 m. This approximate, the October 27, 1982 flood tide situation. As expected, in spite of the narrow opening of the harbor, the small tidal amplitude generates only very weak currents. Current velocities (typical computer output) for $t = 5,580$ and $27,900$ seconds are shown in Figures 4 and 5.

Field data were also obtained for the October 27, 28 flood tide situation. Calibration of the model has not yet been attempted but is expected to be difficult because of the low current velocities predicted by the model and the influence of wind on drift drogues observed in the field.

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